

Existence and Computation of Epistemic EFX Allocations*

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Abstract

We consider the problem of allocating indivisible goods among n agents in a fair manner. For this problem, one of the best notions of fairness is *envy-freeness up to any good* (EFX). However, it is not known if EFX allocations always exist. Hence, several relaxations of EFX allocations have been studied. We propose another relaxation of EFX, called *epistemic EFX* (EEFX). An allocation is EEFX iff for every agent i , it is possible to shuffle the goods of the other agents such that agent i does not envy any other agent up to any good. We show that EEFX allocations always exist for additive valuations, and give a polynomial-time algorithm for computing them. We also show how EEFX is related to some previously-known notions of fairness.

1 Introduction

Fairly dividing a set of goods (or chores) among agents is a fundamental problem in various multi-agent settings that has received a lot of attention in the last decade. The exact definition of *fair* is open to interpretation, and consequently, many different notions of fairness have been studied. In the case where the goods are divisible (i.e., each good can be split into parts and the parts can be distributed among multiple agents), *envy-freeness* and *proportionality* are two very popular notions of fairness. An allocation is called envy-free if every agent values her own bundle more than the bundle of any other agent. An allocation among n agents is called proportional if every agent's value for her bundle is at least $1/n$ times the total value of all goods. In the discrete setting, where goods are indivisible, these notions of fairness are not always achievable, e.g., when there are two agents and a single valuable good, one agent must get nothing. Hence, relaxations of these notions have been explored.

A promising relaxation of envy-freeness is *envy-freeness up to any good* (EFX) [15]. An agent i is said to be *EFX-satisfied* by an allocation if she does not envy any other agent j when any good is removed from j 's bundle. An allocation is called EFX if all agents are EFX-satisfied by that allocation. Formally, let there be n agents and let M be the set of goods. Let $v_i : 2^M \mapsto \mathbb{R}_{\geq 0}$ be agent i 's valuation function. Then agent i is said to be EFX-satisfied by the allocation (X_1, X_2, \dots, X_n) if

$$v_i(X_i) \geq \max_{j \neq i} \max_{g \in X_j} v_i(X_j \setminus \{g\}).$$

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Despite significant effort (according to [15]), it is not yet known if an EFX allocation always exists, even for additive valuations. Hence, there has been significant interest in studying notions of fairness that are relaxations of EFX, like *approximate EFX* [24, 16, 2], and *EFX-with-charity* [14, 19, 18]. *Envy-freeness up to one good* (EF1) [13] is a relaxation of EFX that was studied before EFX. While the question of EFX allocations is still being actively worked on, such relaxations not only shed light on possible approaches to resolve the question, but also give us notions of fairness to use if it is eventually found that EFX allocations do not exist. Motivated by this line of work, we study another relaxation of EFX, called *epistemic EFX* (EEFX), which is fundamentally different from existing relaxations. Before we formally define EEFX, we would like to first give additional motivation.

An important philosophical question is that which of envy-freeness and proportionality is a more appropriate notion of fairness. This question was sidestepped for the fair division of divisible goods since envy-freeness implies proportionality for subadditive valuations. However, for fair division of indivisible goods, this question resurfaces since obtaining EFX allocations has been an elusive goal. This gives an important reason to also consider relaxations of proportionality.

One of the main philosophical arguments in favor of proportionality, as opposed to envy-freeness, is that as long as an agent is getting at least her *due share*, she shouldn't care if some other agent is getting more than her; what others get is none of her business. On a more pragmatic note, it may sometimes be impossible or inappropriate for an agent to know the bundles of the other agents if agents expect others to respect their privacy. This could prevent an agent from verifying whether the allocation is fair if the notion of fairness is based on a relaxation of envy-freeness. However, relaxations of proportionality would not suffer from this problem since an agent can compute her due share without knowing others' bundles.

We now formally define EEFX. Agent i is said to be *EEFX-satisfied* by an allocation (X_1, \dots, X_n) if there is another allocation (Y_1, \dots, Y_n) such that $Y_i = X_i$ and agent i is EFX-satisfied by Y . An allocation is EEFX if every agent is EEFX-satisfied by the allocation. Note that agent i doesn't need to know the bundles of other agents to know that X is a fair allocation; she only needs to know Y , which serves as a *certificate* that X is indeed a fair allocation. It's important to note that different agents can receive different certificates. Essentially, EEFX is a relaxation of proportionality that is defined using ideas from envy-freeness.

Another reason to consider EEFX allocations is that allocations can be *unstable* with respect to EFX. Consider an EFX allocation (X_1, X_2, \dots, X_n) and an agent i . Agents other than i can possibly exchange goods among themselves to increase their utility (see Example 1 for an example). Let (Y_1, \dots, Y_n) be the resulting allocation after this exchange. Note that $Y_i = X_i$, since agent i didn't participate in the exchange. Now agent i need not be EFX-satisfied by Y , but she will be EEFX-satisfied by Y , with X as the certificate. Hence, even if we somehow compute an EFX allocation to begin with, we cannot guarantee that the eventual allocation will be EFX. Given that obtaining EFX allocations has been a difficult open problem, computing an EEFX allocation is a reasonable goal, since EFX allocations may eventually change into EEFX allocations anyways. A Pareto-optimal allocation would prevent agents from exchanging goods among themselves, but it is not known (for non-zero additive valuations) if there exist allocations that are both Pareto-optimal and EFX.

Table 1: An allocation of 9 goods among 3 agents, where the agents have positive, additive, non-identical valuations.

x	a_1	a_2	b_1	b_2	c_1	c_2	d_1	d_2	e
$v_1(x)$	50	50	1	1	10	10	1	1	1
$v_2(x)$	1	1	10	10	1	1	50	50	1
$v_3(x)$	10	10	1	1	10	10	1	1	25

Example 1. Consider the fair-division instance given by Table 1.

Then $X = (\{a_1, a_2, b_1, b_2\}, \{c_1, c_2, d_1, d_2\}, \{e\})$ is an envy-free (and hence EFX) allocation. Let $Y = (\{a_1, a_2, c_1, c_2\}, \{b_1, b_2, d_1, d_2\}, \{e\})$ be the allocation obtained when agents 1 and 2 exchange the goods $\{b_1, b_2\}$ and $\{c_1, c_2\}$. Note that $v_1(Y_1) = 120 > v_1(X_1) = 102$ and $v_2(Y_2) = 120 > v_2(X_2) = 102$, so agents 1 and 2 experience an increase in utility. However,

$$\min_{g \in Y_1} v_3(Y_1 - \{g\}) = 30 > 25 = v_3(Y_3),$$

so agent 3 envies agent 1 even if we remove any good from agent 1. Hence, Y is not an EF1 allocation (and hence also not EFX).

1.1 Our Contribution

We show that EEFX allocations always exist for additive and more generally, for non-negative and cancelable valuations (see Section 2.1 for definition). We also show that EEFX-certificates of all agents can be computed in polynomial-time. We use Barman and Krishnamurthy’s algorithm [8] with a new and involved analysis, and hence our EEFX allocation is also a 2/3-MMS allocation for additive valuations.

An EFX allocation exists when valuations are *ordered* (c.f. Definition 2 in Section 3) [8]. In many fair division contexts, valuations need not be ordered, but there may still be a strong correlation between agents’ valuations. We show (in Section 3.1) that our algorithm outputs an approximate-EFX allocation when valuations are correlated.

In Section 4.1, we prove that for strongly monotone valuations, an MMS allocation is also an EEFX allocation, and that an EEFX allocation is a 4/7-MMS allocation. In Section 4.2, we show that an EEFX allocation is also PROP1, but may not be PROPm (see Section 1.2 for definitions of these notions).

1.2 Related Work

Epistemic Envy-Freeness. The notion of *epistemic envy-freeness* (EEF) was defined by Aziz et al. [3]. EEF allocations may not always exist, e.g., if there are two agents and a single item. [3] shows (for general monotone valuations) that an envy-free allocation is EEF, and that an EEF allocation is a proportional allocation.

MMS. Budish [13] defined a relaxation of proportionality called *maximin share* (MMS). When dividing goods among n agents, the maximin share of agent i , denoted as MMS_i , is defined as the maximum utility agent i can obtain by partitioning the goods into n bundles and picking the bundle with the minimum utility. An allocation is called MMS if each agent i receives a bundle of value at least MMS_i . MMS allocations may not exist [25, 21], even for 3 agents with additive valuations. An allocation is called α -MMS (where $0 < \alpha \leq 1$) if each agent i receives a bundle of value at least $\alpha \cdot \text{MMS}_i$. Barman and Krishnamurthy [8] give an algorithm to compute a $2n/(3n - 1)$ -MMS allocation, where n is the number of agents. Garg and Taki [20] give an algorithm that computes a $(3/4 + 1/(12n))$ -MMS allocation.

EFX allocations in special cases. EFX allocations can be obtained in special cases. Plaut and Roughgarden [24] gave algorithms for finding EFX+MMS+PO allocations for general monotone valuations when there are only 2 agents or when all agents have the same valuation function. A fair division instance consisting of goods $\{g_1, g_2, \dots, g_m\}$ is called *ordered* iff for each agent i , we have $v_i(\{g_1\}) \geq v_i(\{g_2\}) \geq \dots \geq v_i(\{g_m\})$. There is a polynomial-time algorithm for computing an EFX allocation for ordered instances (Lemma 3.5 in [8], Theorem 6.2 in [24]) Chaudhary, Garg and Mehlhorn [17] showed that EFX allocations exist for 3 agents. EFX allocations always exist when the number of goods is at most 2 more than the number of agents [2].

EF1. Budish [13] introduced the notion of *envy-free up to one good* (EF1). An allocation is EF1 iff for every pair (i, j) of agents, it's possible to remove a good from j 's bundle so that i values her own bundle more than j 's bundle. EF1 allocations can be computed in polynomial time [22], even for general monotone valuations. For additive valuations, EF1 allocations that are Pareto-optimal can be computed in pseudopolynomial time [9], and allocations that maximize the Nash Social Welfare (NSW), defined as the geometric mean of agents' valuations, are EF1 [15].

α -EFX. An allocation (X_1, X_2, \dots, X_n) is called α -EFX ($\alpha \leq 1$) iff for every agent i ,

$$v_i(X_i) \geq \alpha \cdot \max_{j \neq i} \max_{g \in X_j} v_i(X_j \setminus \{g\}).$$

Plaut and Roughgarden [24] show how to find 1/2-EFX allocations for subadditive valuations. [16] gives a polynomial-time algorithm to find an allocation that is both EF1 and 1/2-EFX for subadditive valuations. For additive valuations, an NSW-maximizing allocation is 0.6180-EFX [15]. [2] shows how to compute a 0.6180-EFX allocation in polynomial time.

EFX with charity. [14] introduced the notion of EFX-with-charity, i.e., we allow throwing away some goods and allocating the remaining items such that the resulting allocation is EFX. [19] shows how to obtain an EFX allocation for general monotone valuations after throwing away a set P of goods such that no agent envies P and $|P| < n$, where n is the number of agents. [18] shows how to obtain a $(1 - \varepsilon)$ -EFX allocation after throwing away at most $64(n/\varepsilon)^{4/5}$ goods.

PMMS and GMMS. An allocation (X_1, X_2, \dots, X_n) satisfies the α -pairwise maximin share (α -PMMS) guarantee [15] iff for every pair (i, j) of agents, the allocation (X_i, X_j) of the goods $X_i \cup X_j$ among agents i and j is an α -MMS allocation. It can be shown that an α -PMMS allocation is also an α -EFX allocation, even for general monotone valuations. Extending this notion to all k -tuples of agents gives us the α -groupwise maximin share (α -GMMS) notion [6]. Note that an α -GMMS allocation is both an α -PMMS allocation and an α -MMS allocation. [6] mentions many shortcomings of MMS allocations that GMMS allocations don't have. An NSW-maximizing allocation is 0.6180-PMMS. [15]. 1/2-GMMS allocations can be computed in polynomial time [6]. [19] gave an algorithm for computing 4/7-GMMS allocations. 1-GMMS allocations always exist when the number of goods is at most 2 more than the number of agents [2].

PROPx, PROP1, PROPM. PROPx, PROP1 and PROPM are relaxations of proportionality. An allocation $X = (X_1, \dots, X_n)$ of goods M is said to be PROPx if for every agent i , $v_i(X_i) \geq v_i(M)/n - \min_{g \in M \setminus X_i} v_i(\{g\})$ [5, 4]. PROPx allocations do not always exist [4]. An allocation X is said to be PROP1 if for every agent i , $v_i(X_i) \geq v_i(M)/n - \max_{g \in M \setminus X_i} v_i(\{g\})$ [5, 4]. It is easy to see that an EF1 allocation is also a PROP1 allocation. There exists a strongly polynomial-time algorithm for computing allocations that are both PROP1 and Pareto-optimal [7, 4, 23]. An allocation $X = (X_1, \dots, X_n)$ of goods M is called PROPM if for every agent i ,

$$v_i(X_i) \geq \frac{v_i(M)}{n} - \max_{j \neq i} \min_{g \in X_j} v_i(\{g\}).$$

PROPM allocations always exist and can be computed in polynomial time [5].

2 Notation and Preliminaries

For a non-negative integer n , define $[n] := \{1, 2, \dots, n\}$.

In the fair division problem, we are given a set $[m]$ of items that need to be distributed among n agents. An allocation is defined to be a tuple $X := (X_1, X_2, \dots, X_n)$, where $X_i \cap X_j = \emptyset$ for all $i \neq j$ and $\cup_{i=1}^n X_i = [m]$. Here X_i is the set of items allocated to agent i . X_i is called agent i 's *bundle*.

The value that an agent i assigns to a set of items is captured by her *valuation function*, which takes as input a subset of items from $[m]$ and returns a real number. For any valuation function v , we assume that $v(\emptyset) = 0$. For an item g , we will write $v(g)$ instead of $v(\{g\})$ for simplicity. An agent's *disutility function* is the negative of her valuation function.

A function $v : 2^{[m]} \mapsto \mathbb{R}$ is called non-negative if $v(S) \geq 0$ for any $S \subseteq [m]$. v is called monotone if $v(S) \leq v(T)$ whenever $S \subseteq T$. Unless specified otherwise, all valuation functions are assumed to be monotone. v is called additive if $v(S) = \sum_{g \in S} v(\{g\})$ for all $S \subseteq [m]$.

A fair-division instance is given by a list $V := [v_1, v_2, \dots, v_n]$, where $v_i : 2^{[m]} \mapsto \mathbb{R}$ is the i^{th} agent's valuation function, the set of goods is $[m]$ and the set of agents is $[n]$. When valuations are additive, V can be represented as an m -by- n matrix.

Definition 1. For a valuation function v , define another function $w : 2^{[m]} \mapsto \mathbb{R}$ as $w(X) := \max_{g \in X} v(X \setminus \{g\})$ if $X \neq \emptyset$ and let $w(\emptyset) := 0$. Then w is called the *one-less-function* of v , and is denoted by $\text{oneLess}(v)$.

2.1 Cancelable Valuations

A function $v : 2^{[m]} \mapsto \mathbb{R}$ is called *cancelable* if $\forall S \subseteq [m], \forall T \subseteq [m]$ and $\forall g \in [m] \setminus (S \cup T)$, we have $v(S \cup \{g\}) > v(T \cup \{g\}) \implies v(S) > v(T)$ (Definition 2.1 in [10]). The following results are easy corollaries of the definition.

Claim 1. Let v be a cancelable function. Then $\forall S \subseteq [m], \forall T \subseteq [m], \forall R \subseteq [m] \setminus (S \cup T)$, we have $v(S) \geq v(T) \implies v(S \cup R) \geq v(T \cup R)$.

Claim 2. If v is cancelable, then $-v$ is also cancelable.

Cancelable functions generalize additive functions.

2.2 Ordered Valuations

Given a valuation function $v : 2^{[m]} \mapsto \mathbb{R}$, we will see how to construct another valuation function $v' : 2^{[m]} \mapsto \mathbb{R}$, called the *ordered valuation* of v , such that $\{v(1), v(2), \dots, v(m)\} = \{v'(1), v'(2), \dots, v'(m)\}$ and $v'(1) \geq v'(2) \geq \dots \geq v'(m)$. Such valuation functions have been studied in [12, 8].

Definition 2 (Ordered valuation). Let $v : 2^{[m]} \mapsto \mathbb{R}$ be a valuation function. Let $\pi := [\pi_1, \pi_2, \dots, \pi_m]$ be a permutation of $[m]$ such that $v(\pi_1) \geq v(\pi_2) \geq \dots \geq v(\pi_m)$ (break ties by picking smaller items first, i.e., if $v(i) = v(j)$ and $i < j$, then i appears before j in the sequence π). For $S \subseteq [m]$, define $v'(S) := v(\{\pi_i : i \in S\})$. Then v' is called the *ordered valuation* of v , and is denoted by $\text{ordered}(v)$.

A fair division instance is called an *ordered instance* if for every agent i , their valuation function v_i satisfies $v_i = \text{ordered}(v_i)$. Intuitively, an instance is ordered iff all agents value items in the same order.

3 Finding EEFX Allocations for Cancelable Valuations

We will reduce the problem of computing an EEFX allocation to the problem of computing an EFX allocation for an *ordered* instance (recall Definition 2). A similar reduction was used by Barman and Krishnamurthy [8], and Bouveret and Lemaître [12] for the fair division of goods.

Definition 3 (Allocation vector). *Let $X := (X_1, \dots, X_n)$ be an allocation of items $[m]$ among n agents. Let a_j be the agent that received item j in allocation X (i.e., $a_j = i \iff j \in X_i$). Then the list $[a_1, a_2, \dots, a_m]$ is called the allocation vector of X .*

Algorithm 1 gives the outline of our algorithm. This algorithm works in two stages. In the first stage, it computes an allocation X' of items $[m]$ that is EFX according to valuations $[v'_1, \dots, v'_n]$, where $v'_i = \text{ordered}(v_i)$. Let $A := [a_1, \dots, a_m]$ be the allocation vector of X' . The second stage of the algorithm, which we call `pickByList`, has m rounds, where in the t^{th} round, we ask agent a_t to pick their most-valuable unallocated item. See Algorithm 2 for a more formal description of `pickByList`.

Algorithm 1 `BarKri(V)`: Allocates items $[m]$ among n agents. Here $V := [v_1, v_2, \dots, v_n]$, where v_i is the valuation function of agent i . Returns an allocation $X := (X_1, X_2, \dots, X_n)$.

- 1: For each $i \in [n]$, let $v'_i := \text{ordered}(v_i)$.
 - 2: Compute an allocation X' of items $[m]$ that is EFX for valuations $[v'_1, v'_2, \dots, v'_n]$.
 - 3: Let A be the allocation vector of X' .
 - 4: **return** `pickByList(A, V)`.
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Algorithm 2 `pickByList(A, V)`: Allocates items $[m]$ to n agents. Here $A := [a_1, \dots, a_m]$ and $V := [v_1, \dots, v_n]$, where $a_j \in [n]$ and v_i is the valuation function of the i^{th} agent. Returns an allocation $X := (X_1, X_2, \dots, X_n)$.

- 1: Initialize the set $S = [m]$.
 - 2: Initialize $X_i = \emptyset$ for $i \in [n]$.
 - 3: **for** t from 1 to m **do**
 - 4: Let agent a_t pick their favorite item \hat{g} from S , i.e., $\hat{g} \in \arg\max_{g \in S} v_{a_t}(g)$.
 - 5: $X_{a_t} = X_{a_t} \cup \{\hat{g}\}$. $S = S \setminus \{\hat{g}\}$.
 - 6: **end for**
 - 7: **return** (X_1, X_2, \dots, X_n) .
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Theorem 3. *There is a polynomial-time algorithm to find an EFX allocation for an ordered fair division instance when valuations are non-negative and cancelable.*

Proof. When valuations are additive and non-negative, the envy-cycle-elimination algorithm finds an EFX allocation for ordered instances (Lemma 3.5 in [8]). This result can be easily extended to non-negative cancelable valuations by the following observation.

Let $S := \{g_1, g_2, \dots, g_k\}$. For any agent i having valuation function v_i , we can assume without loss of generality that $v_i(g_1) \geq v_i(g_2) \geq \dots \geq v_i(g_k)$, since the fair division instance is ordered. For any $j \in [k-1]$, let $T := S \setminus \{g_j, g_k\}$. By Claim 1, we get that

$$v_i(g_j) \geq v_i(g_k) \implies v_i(T \cup \{g_j\}) \geq v_i(T \cup \{g_k\}) \implies v_i(S \setminus \{g_k\}) \geq v_i(S \setminus \{g_j\}). \quad \square$$

Define $A[= i]$ to be the sequence of rounds (in increasing order) in which agent i gets to pick an item, i.e., $A[= i] := [t \in [m] : a_t = i]$. Similarly, define $A[\neq i] := [t \in [m] : a_t \neq i]$. Since A is the allocation vector of X' , we get that for every agent i , $X'_i = A[= i]$ and $[m] \setminus X'_i = A[\neq i]$.

Definition 4. Let B and C be two sets of items where $|B| = |C|$. Let v be a valuation function. Assume without loss of generality that $B := \{b_1, b_2, \dots, b_k\}$ and $C := \{c_1, c_2, \dots, c_k\}$, where $v(b_1) \geq v(b_2) \geq \dots \geq v(b_k)$ and $v(c_1) \geq v(c_2) \geq \dots \geq v(c_k)$. Then we say that B dominates C in v (denoted as $B \succeq_v C$) if $v(b_j) \geq v(c_j)$ for every $j \in [k]$.

We will now prove two crucial theorems (Theorems 4 and 5) about `pickByList`. Note that these theorems work for arbitrary valuation functions (even ones that are non-monotone or non-positive).

Theorem 4. Let X' be any allocation of items $[m]$ among n agents. Let A be the allocation vector of X' . Let $X := \text{pickByList}(A, V)$, where $V := [v_1, \dots, v_n]$. Then for every $i \in [n]$, $X_i \succeq_{v_i} X'_i$.

Proof. Let $A[= i] := [t_1, t_2, \dots, t_k]$. Let g_j be the j^{th} item picked by agent i , so $X_i = \{g_1, g_2, \dots, g_k\}$. Without loss of generality, assume $v_i(1) \geq v_i(2) \geq \dots \geq v_i(m)$. Thus, agent i always picks the smallest-numbered item.

Let S be the unallocated items available at the beginning of round t_j . At this time, $t_j - 1$ items have already been picked, so $|S| = m - (t_j - 1)$. Agent i picks item $\min(S)$, and $\min(S) \leq t_j$. Hence, $g_j \leq t_j$, which implies $v_i(g_j) \geq v_i(t_j)$, which implies $X_i \succeq_{v_i} A[= i] = X'_i$. \square

Theorem 5. Let X' be any allocation of items $[m]$ among n agents. Let A be the allocation vector of X' . Let $X := \text{pickByList}(A, V)$, where $V := [v_1, \dots, v_n]$. Then for every $p \in [n]$, $[m] \setminus X_p \preceq_{v_p} [m] \setminus X'_p$.

Proof. Without loss of generality, assume $p = 1$ and $v_1(1) \geq v_1(2) \geq \dots \geq v_1(m)$. We say that an item i is smaller than an item j if $i < j$. Let $A[\neq 1] := [t_1, t_2, \dots, t_q]$. Let $[m] \setminus X_1 := [g_1, g_2, \dots, g_q]$, where $g_1 < g_2 < \dots < g_q$. We will prove that $g_i \geq t_i$ for each $i \in [q]$. That would imply $v_1(g_i) \leq v_1(t_i)$, which in turn would imply $[m] \setminus X_p \preceq_{v_p} A[\neq p] = [m] \setminus X'_p$.

The items $[m] \setminus X_1$ may not be picked in the order $[g_1, g_2, \dots, g_q]$, since agents other than agent 1 may have a different preference order over them. Hence, it's possible for an item g_i to be picked in round t_j in `pickByList`, and j may be larger, smaller or equal to i .

Among $\{g_1, g_2, \dots, g_i\}$, let g_k be the last item to be picked. At least $i - 1$ items were picked before g_k , so g_k was picked in round t_ℓ , where $\ell \geq i$. Since g_k was unpicked in the beginning of round t_ℓ , it was also unpicked in the beginning of round t_i . In the first $t_i - 1$ rounds, agent 1 had picked $t_i - i$ items. Since agent 1 only picks the smallest item available to her in each round, these items are smaller than g_k , and hence they are smaller than g_i . The items $\{g_1, g_2, \dots, g_{i-1}\}$, which were picked by agents other than 1, are also smaller than g_i . Hence, there are $(t_i - 1) + (i - 1) = t_i - 1$ items that are smaller than g_i . Hence, $g_i \geq t_i$. \square

Lemma 6. Let v be a cancelable function. Let B and C be sets of items such that $B \succeq_v C$. Then $v(B) \geq v(C)$.

Proof. Let $B := \{b_1, b_2, \dots, b_k\}$ and $C := \{c_1, c_2, \dots, c_k\}$, where $v(b_1) \geq v(b_2) \geq \dots \geq v(b_k)$ and $v(c_1) \geq v(c_2) \geq \dots \geq v(c_k)$. Since $B \succeq_v C$, we get that $v(b_i) \geq v(c_i)$ for all $i \in [k]$.

Let $Z^{(i)} := \{b_1, b_2, \dots, b_i, c_{i+1}, \dots, c_k\}$, where $0 \leq i \leq k$. Then $B = Z^{(k)}$ and $C = Z^{(0)}$. For $i \in [k]$, let $Y^{(i)} := \{b_1, \dots, b_{i-1}, c_{i+1}, \dots, c_k\}$. By Claim 1,

$$v(b_i) \geq v(c_i) \implies v(Y^{(i)} \cup \{b_i\}) \geq v(Y^{(i)} \cup \{c_i\}) \implies v(Z^{(i)}) \geq v(Z^{(i-1)}).$$

This gives us $v(B) = v(Z^{(k)}) \geq v(Z^{(k-1)}) \geq \dots \geq v(Z^{(1)}) \geq v(Z^{(0)}) = v(C)$. \square

Lemma 7. Let $v : 2^{[m]} \mapsto \mathbb{R}$ be a cancelable function. Let $w := \text{oneLess}(v)$. Let B and C be sets of items such that $B \succeq_v C$. Then $w(B) \geq w(C)$.

Proof. Let $B := \{b_1, b_2, \dots, b_k\}$ and $C := \{c_1, c_2, \dots, c_k\}$, where $v(b_1) \geq v(b_2) \geq \dots \geq v(b_k)$ and $v(c_1) \geq v(c_2) \geq \dots \geq v(c_k)$. Since $B \succeq C$, we get that $v(b_i) \geq v(c_i)$ for all $i \in [k]$. The case $k \leq 1$ is trivially true, so assume $k \geq 2$.

For any $j \in [k-1]$, let $T := B \setminus \{b_j, b_k\}$. By Claim 1, we get that

$$v_i(b_j) \geq v_i(b_k) \implies v_i(T \cup \{b_j\}) \geq v_i(T \cup \{b_k\}) \implies v_i(B \setminus \{b_k\}) \geq v_i(B \setminus \{b_j\}).$$

Hence, $w(B) = v(B \setminus \{b_k\})$. Similarly, $w(C) = v(C \setminus \{c_k\})$.

Since $B \setminus \{b_k\} \succeq_v C \setminus \{c_k\}$, we get $w(B) \geq w(C)$ by Lemma 6. \square

Lemma 8. *Let $X := \text{BarKri}(V)$, where $V := [v_1, v_2, \dots, v_n]$, v_i is the valuation of agent i , and v_i is non-negative and cancelable. Then X is an EEFX allocation. Furthermore, given X' from line 2 of $\text{BarKri}(V)$, we can compute the EEFX-certificates of all agents in polynomial time.*

Proof. We will show that agent 1 is EEFX-satisfied by X and show how to compute her EEFX-certificate in polynomial time. Then Lemma 8 would follow by symmetry. Assume without loss of generality that $v_1(1) \geq v_1(2) \geq \dots \geq v_1(m)$. Then $v_1 = v'_1$, where $v'_i := \text{ordered}(v_i)$.

Define $Y_1 := X_1$. Let $[m] \setminus X'_1 := \{t_1, t_2, \dots, t_q\}$. Let $[m] \setminus X_1 := \{g_1, g_2, \dots, g_q\}$. Without loss of generality, assume $v_1(t_1) \geq v_1(t_2) \geq \dots \geq v_1(t_q)$ and $v_1(g_1) \geq v_1(g_2) \geq \dots \geq v_1(g_q)$. For $i \geq 2$, define $Y_i := \{g_j : t_j \in X'_i\}$. Define $Y := (Y_1, Y_2, \dots, Y_n)$. Note that Y can be computed in polynomial time. By Theorem 5, we get $[m] \setminus X_1 \preceq_{v_1} [m] \setminus X'_1$, which implies $v_1(g_j) \leq v_1(t_j)$ for all j . Hence, for every $i \geq 2$, we get $Y_i \preceq_{v_1} X'_i$.

Let $w_i := \text{oneLess}(v_i)$. By Theorem 4, we have $X_1 \succeq_{v_1} X'_1$. By Lemma 6, we get $v_1(X_1) \geq v_1(X'_1)$. X' is an EFX allocation for valuations $[v'_1, \dots, v'_n]$. Since $v_1 = v'_1$, agent 1 is EFX-satisfied by X' for valuation v_1 . This gives us $v_1(X'_1) \geq w_1(X'_1)$ for all $i \in [n]$. Since $X'_i \succeq_{v_1} Y_i$, Lemma 7 gives us $w_1(X'_i) \geq w_1(Y_i)$ for all $i \in [n] \setminus \{1\}$. Hence, for each $i \in [n] \setminus \{1\}$, we get $v_1(Y_1) = v_1(X_1) \geq v_1(X'_1) \geq w_1(X'_i) \geq w_1(Y_i)$. Hence, agent 1 is EFX-satisfied by Y . Therefore, agent 1 is EEFX-satisfied by X and Y is the corresponding certificate. \square

3.1 Partially Ordered Instances

We saw that Algorithm 1 guarantees us an EEFX allocation, but when agents' valuations are ordered, the output is an EFX allocation. In many fair division contexts, valuations need not be ordered. However, we often expect valuations to have some correlation. We show that in this case, Algorithm 1's output is approximately EFX.

Definition 5 (Correlated valuations). *Non-negative valuation functions $\{v_1, v_2, \dots, v_n\}$ are said to be α -correlated (where $\alpha \in [0, 1]$) if $\forall i \neq j$ and $\forall S, T \subseteq [m]$, we have $v_i(S) \geq v_i(T) \implies v_j(S) \geq \alpha \cdot v_j(T)$.*

Definition 6. *Let B and C be two sets of items where $|B| = |C|$. Let v be a non-negative valuation function. Assume without loss of generality that $B := \{b_1, b_2, \dots, b_k\}$ and $C := \{c_1, c_2, \dots, c_k\}$, where $v(b_1) \geq v(b_2) \geq \dots \geq v(b_k)$ and $v(c_1) \geq v(c_2) \geq \dots \geq v(c_k)$. For $\alpha \geq 0$, we say that $B \succeq_v \alpha C$ if $v(b_j) \geq \alpha v(c_j)$ for every $j \in [k]$.*

Lemma 9. *Let $V := [v_1, \dots, v_n]$ be non-negative α -correlated valuations. Let X' be an allocation of items $[m]$ among n agents. Let $X := \text{pickByList}(A, V)$, where A is the allocation vector of X' . Then for any $r \neq s$, we have $X'_s \succeq_{v_r} \alpha X_s$.*

Proof. Without loss of generality, let $r = 1$ and $v_1 = \text{ordered}(v_1)$. Let $X'_s := \{t_1, \dots, t_q\}$ and $X_s := \{g_1, \dots, g_q\}$, where $v_1(t_1) \geq \dots \geq v_1(t_q)$ and $v_1(g_1) \geq \dots \geq v_1(g_q)$. It is sufficient to prove that $v_1(t_k) \geq \alpha v_1(g_k)$ for all $k \in [q]$.

Let S be the set of unallocated items at the beginning of round t_k of `pickByList`. Let $i := \min(S)$ and $j := \max([m] \setminus S)$, i.e., i is the first unallocated item and j is the last allocated item. Since g_k got allocated in round t_k , we have $g_k \geq i$.

Case 1: $j < i$. Since $t_k - 1$ items have been allocated so far, we have $j = t_k - 1$ and $i = t_k$. Hence, $g_k \geq i = t_k$, so $v_1(t_k) \geq v_1(g_k) \geq \alpha v_1(g_k)$.

Case 2: $i > j$. Since $t_k - 1$ items have been allocated so far, we have $i \leq t_k - 1$ and $j \geq t_k$. Since $j \notin S$, it was picked by someone in an earlier round. Suppose agent p had picked j in round $\ell < t_k$. Since $i \in S$, i was unallocated in round ℓ . Since agent p picked j instead of i , we get $v_p(j) \geq v_p(i)$. Since valuations are α -correlated, we get $v_1(j) \geq \alpha v_1(i)$. Hence, $v_1(t_k) \geq v_1(j) \geq \alpha v_1(i) \geq \alpha v_1(g_k)$.

Hence, $v_1(t_k) \geq \alpha v_1(g_k)$ for all $k \in [q]$. \square

Lemma 10. *Let $X := \text{BarKri}(V)$, where $V := [v_1, v_2, \dots, v_n]$, v_i is the valuation of agent i , and v_i is non-negative and additive. If valuations V are α -correlated, then X is an α -EFX allocation.*

Proof. Let $v'_i := \text{ordered}(v_i)$ for $i \in [n]$. Let X' be an EFX allocation for valuations $[v'_1, \dots, v'_n]$ (as computed at Line 2 of `BarKri`). Let A be the allocation vector of X' . Then $X = \text{pickByList}(A, V)$.

Let $w_i := \text{oneLess}(v_i)$. We will show that $v_1(X_1) \geq \alpha w_1(X_i)$ for every $i \geq 2$. By symmetry, this would imply that X is α -EFX. Assume without loss of generality that $v_1 = v'_1$. Since X' is EFX for valuations $[v'_1, \dots, v'_n]$, we get that agent 1 is EFX-satisfied by X' .

By Lemma 9, $X'_i \succeq_{v_1} \alpha X_i$ for $i \in [n] \setminus \{1\}$, so $w_1(X'_i) \geq \alpha w_1(X_i)$. Hence,

$$\begin{aligned} v_1(X_1) &\geq v_1(X'_1) && \text{(by Theorem 4)} \\ &\geq w_1(X'_i) && \text{(agent 1 is EFX-satisfied by } X') \\ &\geq \alpha w_1(X_i). \end{aligned}$$

Hence, X is α -EFX. \square

3.2 Chores

In this section, we see how to extend our results so far to fair-division of chores.

Theorem 11. *There is a polynomial-time algorithm to find an EFX allocation for an ordered fair division instance when valuations are non-positive and cancelable.*

Proof sketch. When valuations are non-positive and cancelable, we can show that the top-trading-envy-cycle-elimination algorithm [11] finds an EFX allocation for ordered instances when items are allocated in non-increasing order of disutility. \square

Lemma 12. *Let $X := \text{BarKri}(V)$, where $V := [v_1, v_2, \dots, v_n]$, v_i is the valuation of agent i , and v_i is non-positive and cancelable. Then X is an EEFX allocation. Furthermore, given X' from line 2 of `BarKri}(V), we can compute the EEFX-certificates of all agents in polynomial time.`*

Proof. We will show that agent 1 is EEFX-satisfied by X and show how to compute her EEFX-certificate in polynomial time. Then Lemma 12 would follow by symmetry. Assume without loss of generality that $v_1(1) \geq v_1(2) \geq \dots \geq v_1(m)$. Then $v_1 = v'_1$, where $v'_i := \text{ordered}(v_i)$. Let $d_i := -v_i$ and $d'_i := -v'_i$.

Define $Y_1 := X_1$. Let $[m] \setminus X'_1 := \{t_1, t_2, \dots, t_q\}$. Let $[m] \setminus X_1 := \{c_1, c_2, \dots, c_q\}$. Without loss of generality, assume $d_1(t_1) \leq d_1(t_2) \leq \dots \leq d_1(t_q)$ and $d_1(c_1) \leq d_1(c_2) \leq \dots \leq d_1(c_q)$. For

$i \geq 2$, define $Y_i := \{c_j : t_j \in X'_i\}$. Define $Y := (Y_1, Y_2, \dots, Y_n)$. Note that Y can be computed in polynomial time. By Theorem 5, we get $[m] \setminus X_1 \preceq_{v_1} [m] \setminus X'_1$, which implies $d_1(c_j) \geq d_1(t_j)$ for all j . Hence, for every $i \geq 2$, we get $Y_i \succeq_{d_1} X'_i$.

Let $w_i := \text{oneLess}(d_i)$. By Theorem 4, we have $X_1 \preceq_{d_1} X'_1$. By Lemma 7, we get $w_1(X_1) \leq w_1(X'_1)$. X' is an EFX allocation for valuations $[v'_1, \dots, v'_n]$. Since $v_1 = v'_1$, agent 1 is EFX-satisfied by X' for valuation v_1 . This gives us $w_1(X'_1) \leq d_1(X'_i)$ for all $i \in [n]$. Since $X'_i \preceq_{d_1} Y_i$, Lemma 6 gives us $d_1(X'_i) \leq d_1(Y_i)$ for all $i \in [n] \setminus \{1\}$. Hence, for each $i \in [n] \setminus \{1\}$, we get $w_1(Y_1) = w_1(X_1) \leq w_1(X'_1) \leq d_1(X'_i) \leq d_1(Y_i)$. Hence, agent 1 is EFX-satisfied by Y . Therefore, agent 1 is EEFX-satisfied by X and Y is the corresponding certificate. \square

Lemma 13. *Let $V := [v_1, \dots, v_n]$ be non-positive valuations. Let X' be an allocation of items $[m]$ among n agents. Let $X := \text{pickByList}(A, V)$, where A is the allocation vector of X' . Let $d_i := -v_i$ for $i \in [n]$, and let $[d_1, \dots, d_n]$ be $(1/\alpha)$ -correlated. Then for any $r \neq s$, we have $X_s \succeq_{d_r} (1/\alpha)X'_s$.*

Proof. Without loss of generality, let $r = 1$ and $v_1 = \text{ordered}(v_1)$. Let $X'_s := \{t_1, \dots, t_q\}$ and $X_s := \{c_1, \dots, c_q\}$, where $d_1(t_1) \leq \dots \leq d_1(t_q)$ and $d_1(c_1) \leq \dots \leq d_1(c_q)$. It is sufficient to prove that $d_1(t_k) \leq \alpha d_1(c_k)$ for all $k \in [q]$.

Let S be the set of unallocated items at the beginning of round t_k of `pickByList`. Let $i := \min(S)$ and $j := \max([m] \setminus S)$, i.e., i is the first unallocated item and j is the last allocated item. Since c_k got allocated in round t_k , we have $c_k \geq i$.

Case 1: $j < i$. Since $t_k - 1$ items have been allocated so far, we have $j = t_k - 1$ and $i = t_k$. Hence, $c_k \geq i = t_k$, so $d_1(t_k) \leq d_1(c_k) \leq \alpha d_1(c_k)$.

Case 2: $i > j$. Since $t_k - 1$ items have been allocated so far, we have $i \leq t_k - 1$ and $j \geq t_k$. Since $j \notin S$, it was picked by someone in an earlier round. Suppose agent p had picked j in round $\ell < t_k$. Since $i \in S$, i was unallocated in round ℓ . Since agent p picked j instead of i , we get $d_p(j) \leq d_p(i)$. Since valuations are α -correlated, we get $d_1(j) \leq \alpha d_1(i)$. Hence, $d_1(t_k) \leq d_1(j) \leq \alpha d_1(i) \leq \alpha d_1(c_k)$.

Hence, $d_1(t_k) \leq \alpha d_1(c_k)$ for all $k \in [q]$. \square

Lemma 14. *Let $X := \text{BarKri}(V)$, where $V := [v_1, v_2, \dots, v_n]$, v_i is the valuation of agent i , and v_i is non-positive and additive. Let $d_i := -v_i$. If $[d_1, \dots, d_n]$ are $(1/\alpha)$ -correlated, then X is an α -EFX allocation.*

Proof. Let $v'_i := \text{ordered}(v_i)$ for $i \in [n]$. Let X' be an EFX allocation for valuations $[v'_1, \dots, v'_n]$ (as computed at Line 2 of `BarKri`). Let A be the allocation vector of X' . Then $X = \text{pickByList}(A, V)$.

Let $w_i := \text{oneLess}(d_i)$. We will show that $w_1(X_1) \leq \alpha d_1(X_i)$ for every $i \geq 2$. By symmetry, this would imply that X is α -EFX. Assume without loss of generality that $v_1 = v'_1$. Since X' is EFX for valuations $[v'_1, \dots, v'_n]$, we get that agent 1 is EFX-satisfied by X' .

By Lemma 13, $X_i \succeq_{d_1} (1/\alpha)X'_i$ for $i \in [n] \setminus \{1\}$, so $d_1(X'_i) \leq \alpha d_1(X_i)$. Hence,

$$\begin{aligned} w_1(X_1) &\leq w_1(X'_1) && \text{(by Theorem 4)} \\ &\leq d_1(X'_i) && \text{(agent 1 is EFX-satisfied by } X') \\ &\leq \alpha d_1(X_i). \end{aligned}$$

Hence, X is α -EFX. \square

4 Relationship Between EEFX and Other Notions of Fairness

4.1 MMS

Theorem 15. *Let $X = (X_1, X_2, \dots, X_n)$ be an allocation of goods $[m]$ among n agents. Let v be the valuation function of agent k and let MMS_k be the maximin share of agent k . Let v be non-negative and strongly monotone, i.e., $\forall Y \subset X \subseteq [m], v(Y) < v(X)$. If $v(X_k) \geq \text{MMS}_k$, then agent k is EEFX-satisfied by X .*

Proof. Without loss of generality, assume $k = 1$. For a non-empty set S of items, define $\text{top}(S) := \arg\max_{g \in S} v(S \setminus \{g\})$, $w(S) := \max_{g \in S} v(S \setminus \{g\})$ and $w(\emptyset) := 0$.

We will obtain an EEFX-certificate of agent 1 for X by modifying $[m] \setminus X_1$. This modification will take place in multiple iterations. For an allocation $Y = (Y_1, \dots, Y_n)$, define

$$\begin{aligned}\phi_1(Y) &:= |\{j : j > 1 \text{ and } v(Y_j) \leq v(Y_1)\}|. \\ \phi_2(Y) &:= \max_{i=2}^n w(Y_i). \\ \phi_3(Y) &:= |\{j : j > 1 \text{ and } w(Y_j) = \phi_2(Y)\}|. \\ \phi(Y) &:= (\phi_1(Y), \phi_2(Y), \phi_3(Y)).\end{aligned}$$

For allocations Y and Z , we say that $\phi(Y) < \phi(Z)$ iff $\phi(Y)$ is lexicographically less than $\phi(Z)$.

In any iteration, if $\phi_2(X) \leq v(X_1)$, then X is an EEFX-certificate for agent 1, and we're done. Otherwise, we will modify $[m] \setminus X_1$ such that $\phi(X)$ reduces. Since $\phi(X)$ can only take finitely many values, we will eventually reach a point where $\phi_2(X) \leq v(X_1)$. Hence, we will assume $\phi_2(X) > v(X_1)$ from now on.

Assume $\phi_1(X) = 0$. Then $v(X_i) > v(X_1)$ for all $i > 1$. Without loss of generality, assume $w(X_n) = \phi_2(X)$. Let $g_n := \text{top}(X_n)$. Let $Y_1 := X_1 \cup \{g_n\}$, $Y_n := X_n - \{g_n\}$, and $Y_j := X_j$ for $j \in [n-1] - \{1\}$. Then $v(Y_n) = w(X_n) = \phi_2(X) > v(X_1)$, $v(Y_1) > v(X_1)$ (by strong monotonicity), and for $j \in [n-1] - \{1\}$, $v(Y_j) = v(X_j) > v(X_1)$. Hence, $\min_{i=1}^n v(Y_i) > v(X_1) > \text{MMS}_1$. This is a contradiction, by the definition of maximin share. Hence, $\phi_1(X) \geq 1$.

Without loss of generality, assume $w(X_n) := \phi_2(X)$ and $v(X_2) \leq v(X_1)$. Let $g_n := \text{top}(X_n)$. Let $Y_2 := X_2 \cup \{g_n\}$, $Y_n := X_n - \{g_n\}$, and $Y_j := X_j$ for $j \notin \{2, n\}$. Then $v(Y_n) = w(X_n) > v(X_1)$. Hence, $\phi_1(Y) \leq \phi_1(X)$. If $v(Y_2) > v(X_1)$, then $\phi_1(Y) = \phi_1(X) - 1$, and we replace X by Y and this iteration ends.

Now let $v(Y_2) \leq v(X_1)$. Then $\phi_1(Y) = \phi_1(X)$. Then $w(Y_2) \leq v(Y_2) \leq v(X_1) < w(X_n) = \phi_2(X)$. Also, $w(Y_n) < v(Y_n) = w(X_n) = \phi_2(X)$ (by strong monotonicity). Hence, $\phi(Y) < \phi(X)$, and this iteration ends. \square

Corollary 15.1. *For strongly monotone valuations, an MMS allocation is also an EEFX allocation.*

When valuations are not strongly monotone, an MMS allocation may not be an EEFX allocation, even for identical and additive valuations. Consider the following example (modified from [24]): Let there be 2 goods $\{a, b\}$ and 2 agents. Let $v_1(a) = 0$ and $v_1(b) = 1$. Then $\text{MMS}_1 = 0$, so $(\emptyset, \{a, b\})$ is an MMS allocation. However, agent 1 is not EEFX-satisfied.

Even though MMS implies EEFX, approximate MMS doesn't imply approximate EEFX.

Observation 16. *Let there be 2 agents having identical and additive valuations. Let there be 3 goods of values 1, $\varepsilon(1 - \varepsilon)$, and ε^2 , where $0 < \varepsilon < 1$. Let X be an allocation where agent 1 receives a good of value $\varepsilon(1 - \varepsilon)$ and agent 2 receives the other two goods. Then X is $(1 - \varepsilon)$ -MMS, but not ε -EEFX.*

Lemma 17. Let $X = (X_1, X_2, \dots, X_n)$ be an allocation where agent i is EFX-satisfied. Then for non-negative additive valuations, $v_i(X_i) \geq \frac{4n}{7n-2} \text{MMS}_i$.

Proof. Can be inferred from the proof of Proposition 3.4 in [1]. \square

Corollary 17.1. An EEFX allocation among n agents is also a $4n/(7n-2)$ -MMS allocation under non-negative additive valuations.

Proof. Let Y be an EEFX allocation. For any agent i , let X be the corresponding EEFX-certificate. Then $v_i(Y_i) = v_i(X_i) \geq \frac{4n}{7n-2} \text{MMS}_i$ by Lemma 17. \square

4.2 PROP_x, PROP_m, PROP1

We define a relaxation of EEFX and EF1, called EEF1, and show that an EEF1 allocation is also a PROP1 allocation (c.f. Section 1.2 to recall definitions of PROP_x, PROP_m and PROP1). This implies that MMS allocations (for strongly monotone valuations), EF1 allocations and EEFX allocations are also PROP1 allocations.

On the other hand, we give examples of allocations that are PROP_x but not EEF1. This shows that PROP_x allocations may not be MMS, EF1 or EEFX.

Definition 7. Agent i is said to be EEF1-satisfied by allocation $X = (X_1, \dots, X_n)$ if there exists an allocation $Y = (Y_1, \dots, Y_n)$ such that $Y_i = X_i$ and agent i is EF1-satisfied by Y , i.e.,

$$v_i(Y_i) \geq \max_{j \neq i} \min_{g \in X_j} v_i(X_j \setminus \{g\}).$$

Y is then called the EEF1-certificate of X . Allocation X is called EEF1 if every agent is EEF1-satisfied by X .

Lemma 18. Let X be an EEF1 allocation of items $[m]$ among n agents. When valuations are additive, X is also a PROP1 allocation.

Proof. Let Y be the EEF1-certificate of X for agent i . Then

$$\begin{aligned} v_i(X_i) &= v_i(Y_i) \geq \max_{j \neq i} \min_{g \in Y_j} v_i(Y_j \setminus \{g\}) \\ &\geq \frac{1}{n-1} \sum_{j \neq i} \min_{g \in Y_j} v_i(Y_j \setminus \{g\}) \\ &= \frac{v_i([m] \setminus X_i)}{n-1} - \frac{1}{n-1} \sum_{j \neq i} \max_{g \in Y_j} v_i(g) \\ &\geq \frac{v_i([m] \setminus X_i)}{n-1} - \max_{j \neq i} \max_{g \in Y_j} v_i(g) \\ &= \frac{v_i([m] \setminus X_i)}{n-1} - \max_{g \in [m] \setminus X_i} v_i(g) \\ \implies v_i(X_i) &\geq \frac{v_i([m])}{n} - \frac{n-1}{n} \max_{g \in [m] \setminus X_i} v_i(g) \geq \frac{v_i([m])}{n} - \max_{g \in [m] \setminus X_i} v_i(g). \end{aligned}$$

Hence, X is a PROP1 allocation. \square

Lemma 19. There is a PROP_x allocation of 6 goods among 2 agents that is not an EEF1 allocation, even when agents' valuations are identical, additive and positive.

Proof. When there are only two agents, an allocation is EEF1 iff it is EF1.

Let v be the valuation function of both agents. Let $v(1) = v(2) = v(3) = 1$ and $v(4) = v(5) = v(6) = 2$. Let $X = (\{1, 2, 3\}, \{4, 5, 6\})$. Then $v(X_1) = 3$ and $v([m])/2 - \min_{g \in [m] \setminus X_1} v(g) = 9/2 - 2 = 5/2$. Hence, X is PROPx. However, $\min_{g \in X_2} v(X_2 - \{g\}) = 4$, so X is not EF1. \square

Lemma 20. *There is an MMS allocation of 6 goods among 3 agents that is not a PROPm allocation, even when agents' valuations are identical, additive and positive.*

Proof. Let $X := (X_1, X_2, X_3)$ be an allocation, where X_1 has one item of value 3, X_3 has 3 items of value 1, and X_2 has an item of value 6 and an item of value 1. Then the MMS is 1 but the PROPm threshold for agent 1 is $10/3$. \square

Corollary 20.1. *An EEFX allocation may not be a PROPm allocation.*

Lemma 21. *Let X be an EFX allocation of items $[m]$ among n agents. When valuations are additive, X is also a PROPm allocation.*

Proof. Let $i \in [n]$. Since X is EFX, we get

$$\begin{aligned}
v_i(X_i) &\geq \max_{j \neq i} \max_{g \in X_j} v_i(X_j \setminus \{g\}) \\
&\geq \frac{1}{n-1} \sum_{j \neq i} \max_{g \in X_j} v_i(X_j \setminus \{g\}) \\
&= \frac{v_i([m] \setminus X_i)}{n-1} - \frac{1}{n-1} \sum_{j \neq i} \min_{g \in X_j} v_i(g) \\
&\geq \frac{v_i([m] \setminus X_i)}{n-1} - \max_{j \neq i} \min_{g \in X_j} v_i(g) \\
\implies v_i(X_i) &\geq \frac{v_i([m])}{n} - \frac{n-1}{n} \max_{j \neq i} \min_{g \in X_j} v_i(g) \\
&\geq \frac{v_i([m])}{n} - \max_{j \neq i} \min_{g \in X_j} v_i(g).
\end{aligned}$$

Hence, X is a PROPm allocation. \square

5 Open Problems

1. Can we achieve a better approximation guarantee for maximin share along with EEFX? Currently we achieve $2/3$ -MMS with EEFX. Is it possible to improve it to the best known approximation factor of $3/4$ for MMS?
2. To prove NP-hardness for the following decision problem: Given an allocation for fair-division instance with goods, decide if it is EEFX.
3. Existence and computation of EEFX+EF1 allocations. A counter-example for this question will also provide a counter-example for the existence of EFX allocations.

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